Generating functions for the Coxeter group H_4

1996 J. Phys. A: Math. Gen. 29 7705

(http://iopscience.iop.org/0305-4470/29/23/028)

J. Phys. A: Math. Gen. **29** (1996) 7705–7719. Printed in the UK

Generating functions for the Coxeter group H_4

C S Lam, J Patera† and R T Sharp

Department of Physics, McGill University, Montreal H3W 2T8, Quebec, Canada

Received 23 January 1996, in final form 8 July 1996

Abstract. Generating functions are found for decomposition of the space of homogeneous polynomials of any degree in four variables into the direct sum of subspaces irreducible under the group H_4, the non-crystallographic Coxeter group of order 14 400. The four variables are coordinates of a vector from the defining four-dimensional representation space of the group. As the defining representation we consider any of the four non-equivalent irreducible representations of H_4 of dimension four.

Analogous generating functions for the binary icosahedral group H_3 of order 120 (generated by reflections, i.e. a subgroup of O(3) but not of SO(3)) and for the dihedral group H_2 of order 10 are also rederived and shown. The groups are naturally related by $H_4 \supset H_3 \supset H_2$.

1. Introduction

The group H_4 is the largest of its kind and as such it plays a special role in mathematics. During the last decade its importance in physics applications has also emerged. Let us point out, for example, that it contains as subgroups the icosahedral group, as well as all the other point groups of three-dimensional physics. Thus it is natural to start by asking the most intriguing question: What is its role in nature?

As far as we know the group H_4 has occurred in the physics literature either in the context of the physics of amorphous solids [1–6], biophysics [7], or quasicrystals [8, 9]. In the latter case it appears to play a rather basic role for a family of quasicrystals displaying icosahedral symmetry and also all the other symmetries encountered in quasicrystal experiments [9, 10]. Moreover, one may expect that the role of H_4 goes well beyond these fields because of its close relation to the largest exceptional simple Lie group E_8 [8–11] which itself has been used in fundamental questions of particle theory. Namely, the gauge group of the heterotic string theory is $E_8 \times E_8$ [12].

Basic properties of finite Coxeter groups, H_4 in particular, have been known for some time [13, 14]. They are of two types: the crystallographic and non-crystallographic ones. They are best distinguished by the properties of their root systems. The former are the symmetry groups of the root systems of the semisimple Lie algebras (Weyl groups) and as such they are the point groups of the corresponding root lattices. The non-crystallographic ones are all the others. Namely, the irreducible ones (connected Coxeter diagram) are the three groups $H_2 \subset H_3 \subset H_4$ considered in this paper and the series of dihedral groups of order $2n$, $7 \leqslant n < \infty$. The fact that there is a root system in the usual sense attached to a non-crystallographic Coxeter group was established

† Permanent address: Département de mathématiques, Université de Montréal, Canada.

0305-4470/96/237705+15$19.50

only relatively recently [15]. Integer linear combinations of such roots are dense point sets.

The group H_4 is a subgroup of O(4) which is generated by four reflections acting in the real Euclidean four-dimensional space $\mathbb{R}^4$. Among the finite groups generated by any number of reflections (Coxeter groups) it is singled out because it is not contained in any larger finite irreducible non-crystallographic Coxeter group (among the crystallographic ones it is found only in the Weyl group of E_8 [9, 11] and higher ones). One may rightly compare the position of H_4 among finite Coxeter groups with that of E_8 among the finite-dimensional semisimple Lie algebras/groups. The order of H_4 is 14 400. Besides the one-dimensional representations, it has four inequivalent irreducible representations of dimension four, and 28 other irreducible representations of dimensions ranging from six to 36. Any one of the four-dimensional representations can be chosen as the defining representation for the group; below we denote such a representation by the symbol $\square$.

Our goal in this paper is the following. We fix the representation $\square$ and consider a generic vector $\boldsymbol{X} = (x_1, x_2, x_3, x_4)$, given relative to some basis of $\mathbb{R}^4$, as being transformed by the representation $\square$. The object of our interest is the space $\mathcal{P}_d$ spanned by the homogeneous polynomials in x_1, x_2, x_3, x_4 of any degree d. It is well known that this space is completely reducible into the direct sum of subspaces irreducible under some representations of H_4. Our problem is to decompose $\mathcal{P}_d$ with any $1 \leqslant d < \infty$ into the direct sum of subspaces irreducible with respect to H_4.

At the same time we find the degrees and the H_4 transformation properties of the integrity basis for our problem. These are homogeneous polynomials in x_1, x_2, x_3, x_4 transforming according to some irreducible representation of H_4. Our aim is to determine their existence, degrees in the four variables, and H_4 transformation properties. We do not do this here, but it would be interesting to construct these tensors explicitly relative to some basis of $\mathbb{R}^4$.

An important simplest case is the integrity basis for H_4 invariants or scalars. It is known that the degrees of its four elements are 2, 12, 20, and 30 (cf table on p 87 in [13]). Every other invariant which is a function of polynomials in x_1, x_2, x_3, x_4 is constructed out of them.

An efficient way to find the information about the elements of the integrity basis is to construct the appropriate generating function. We construct it directly in a 'positive' form, characterized by the property that there are no cancellations during its development into power series. The existence of the elements of the integrity basis, their degrees, and transformation properties under H_4 are then readily inferred from the positive form of the generating function.

The main results of this paper are the generating functions for the H_4 polynomial tensors in their positive form. More precisely, the generating function is constructed for each of the four cases where the representation $\square$ is one of the four irreducible representations of H_4 of dimension four. Only one of the generating functions needs to be calculated, the other three are obtained by certain elementary transformations of the first one, which we indicate. It turns out that the H_4 generating function is so large that it is impractical to spell it out explicitly as a rational function. Nevertheless, there is a concise way to provide all the details of the structure of that function in the form of several tables.

As an independently interesting and didactic introduction to our problem and methods, we find analogous generating functions for the two smaller groups closely related to H_4. These are H_3, the icosahedral group of order 120 generated by three reflections, and the group H_2, the dihedral group of order 10.

Generating functions for scalars of finite groups of not very large orders have been known for about a century as the Molien functions [16, 17]. Many explicit examples can be found in the physics literature (see for example [17–23]). McLellan [17] and Michel [20] give a generalization by which generating functions of other irreducible representations might be calculated; this traditional approach is based on the general property of characters, see for example (2) in [21]. Although it would be conceivable to use computer algebra to find the generating functions of this paper in the traditional way, we have followed a different path which is far more rapid and economical. In fact our method would have allowed us to find the generating function of this paper even by hand. The traditional approach found generating functions for each irreducible representation separately; the present method finds them all simultaneously. The generating function here is the sum of the traditional ones.

The method employed in this paper has apparently not been used since its inception [24]. It consists of finding the numerator and denominator of the generating function in the desired form by a recursion process using only the decompositions of several lowest tensor products of irreducible representations. One of the goals of this paper is to demonstrate for a large case such as H_4 how efficient our method really is.

The generating functions for H_2 can be found in [21]; those for the icosahedral subgroup of $SO(3)$ and $SU(2)$ in [21, 22]. An explicit construction of polynomial tensor integrity bases has been undertaken in [21, 22, 25] starting from the three-dimensional representations.

An interesting problem naturally extending this paper, which we do not consider here, is the actual construction of the tensors of the integrity bases as polynomials in x_1, x_2, x_3, x_4. Such a construction for H_k, and particularly in the case of H_4 where some of the degrees of the desired tensors are rather large and the tensor components may have many terms, would appear practical provided one uses a basis in $\mathbb{R}^k$ which is 'adapted' to the symmetry group H_k. That is either the basis of simple roots of H_k or its dual [10, 26].

The next two sections introduce Coxeter groups in general (section 2) and H_4, H_3, and H_2 (section 3). In section 4 the general form of the generating function is given. Crucial for this paper are the recurrence relations (4.4) allowing us to compute the numerator of the generating function. In sections 5, 6 and 7 we deal with the cases H_2, H_3, and H_4, respectively. The character tables are shown and the generating functions are described. Important and instructive are the H_2 and H_3 examples of computing the numerators in sections 5 and 6.

Irreducible representations of H_k are identified and numbered in the corresponding character table. The symbol χ_i denotes the ith irreducible representation. In column 1a of each character table, containing the characters of the identity element of H_k, one finds the dimensions of the representations.

2. Coxeter groups and their diagrams

A Coxeter group W acting in $\mathbb{R}^n$, the n-dimensional real Euclidean space, is generated by its elementary reflections $r_1, r_2, \ldots, r_n$. To each reflection r_k one associates a reflection plane (mirror) and a normalized vector α_k, called the simple root of W, orthogonal to the mirror. In order to define the group W, angles between the simple roots (or between the mirrors) have to be given. Finite Coxeter groups have been classified for all $1 \leqslant n < \infty$.

A standard presentation of a Coxeter group [14] is provided by the identities

$$(r_j r_k)^{m_{jk}} \qquad j, k = 1, \ldots, n \tag{2.1}$$

and by the Coxeter matrix $\mathbf{M} = (m_{jk})$ with positive integer matrix elements. Two reflections, say r_1 and r_2, are orthogonal if they commute, i.e. $(r_1 r_2)^2 = 1$. It follows from (2.1) that the angle between the jth and kth mirrors is π/m_{jk}. The corresponding angle between the normals to the mirrors, the simple roots α_j and α_k, is then $\pi - \pi/m_{jk}$.

A concise way to obtain the matrix $\mathbf{M}$ is to read its matrix elements off the corresponding Coxeter diagram. Coxeter diagrams are drawn using the following conventions.

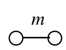
(i) Nodes of the diagram stand for the reflections $r_1, \ldots, r_n$ generating W. The nodes are assumed to be numbered as the reflections. The nodes can also be interpreted either as the mirrors of a kaleidoscope or as suitably normalized vectors $\alpha_1, \ldots, \alpha_n$ (simple roots of W) orthogonal to the mirrors.

(ii) A line linking the jth and kth nodes carries the integer m_{jk} of (2.1). The value $m_{jk} = 3$ is not shown in the diagram. The connecting line between nodes is omitted in the case $m_{jk} = 2$ (orthogonal reflection planes).

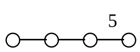
Pertinent examples of Coxeter diagrams are found in (3.1) and (3.2) below.

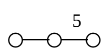
3. The Coxeter groups H_2, H_3, H_4

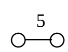
The finite non-crystallographic Coxeter groups are by definition the groups generated by reflections which are *not* the Weyl groups of semisimple Lie algebras. Explicitly, there is the infinite series of dihedral groups (i.e. generated by two reflections, $n = 2$) of order $|\mathcal{D}| = 2m$,

$\mathcal{D}_m (7 \leqslant m < \infty)$  (3.1)

and the three isolated groups

H_4 : 

H_3 : 

H_2 :  (3.2)

of orders 120^2, 120, and 10, respectively. Note that H_2 is the dihedral group and could have been included in (3.1) as the case $m = 5$; H_3 is the (reflection generated) icosahedral group. A close relation between the groups H_4, H_3, and H_2 is conveyed by the similarity of their diagrams, in particular the inclusions

$$H_2 \subset H_3 \subset H_4. \tag{3.3}$$

For practical reasons we are excluding from consideration the Coxeter groups whose generating reflections can be split into two subsets which are pairwise orthogonal. Such groups would have disconnected Coxeter diagrams.

It is known that all finite non-crystallographic Coxeter groups with connected Coxeter diagrams are those listed in (3.1) and (3.2).

Let us recall that the only group generated by a single reflection is the (crystallographic) Coxeter/Weyl group of the simple Lie algebra A_1; its order is 2. Restriction to non-crystallographic groups excludes from (3.1) four dihedral Coxeter/Weyl groups with $m = 2$, 3, 4, and 6. These are the Weyl groups of the semisimple Lie algebras $A_1 + A_1$, A_2, $B_2 \simeq C_2$, and G_2, respectively.

In this paper we are interested in H_4 and its representations. At the same time it is useful and instructive to consider as well its lower dimensional analogues, H_2 and H_3, and their irreducible representations. Basic information about irreducible representations of the three groups is given by their character tables (tables 1, 3, and A1 later).

An explicit description of H_4, H_3, and H_2 is found, for example, in [10, 26]. We recall that the Cartan matrix of H_k, $k = 2, 3, 4$, is defined in the way standard in Lie theory, that is $\mathbf{C}(H_k) := 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ for $1 \leqslant i,\ j \leqslant k$, where $(,)$ denotes the scalar product. Using the conventions implied by the Coxeter diagram, we find readily from (3.2) the three Cartan matrices:

$$\begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix} \qquad \tau = 2\cos\frac{\pi}{5}. \tag{3.4}$$

4. How the generating functions are derived

Our object is the derivation of generating functions for group tensors (i.e. tensors transforming irreducibly under the corresponding group) whose components are polynomials in the variables transforming irreducibly under the defining k-dimensional representation of H_k, $k = 2, 3, 4$. The defining representation is denoted here by $\square$. In this representation the reflection, $r_i (1 \leqslant i \leqslant k)$, generating the corresponding Coxeter group is the $k \times k$ identity matrix with the ith row of the Cartan matrix $\mathbf{C}$ of (3.4) subtracted from the ith column, i.e.

$$(r_i)_{jk} = \delta_{jk} - \delta_{ik} C_{kj}. \tag{4.1}$$

We also give the generating functions for tensors based on other equidimensional representations which are so similar that no extra work is needed to obtain them.

The generating functions we seek will have the form

$$F_\square(\lambda) = \frac{\sum_{p=0}^{p_{\max}} \{p\} \lambda^p}{\prod_{i=1}^{k} (1 - \lambda^{d_i})}. \tag{4.2}$$

Here λ is the dummy variable whose exponent equals the degree of the term it multiplies. In the denominator λ^{d_i} denotes an invariant of degree d_i (there are precisely k of them). In the numerator $\{p\}$ denotes the direct sum of polynomial tensors of degree p; those which contain denominator invariants as factors are excluded:

$$p_{\max} = d_1 + \cdots + d_k - k. \tag{4.3}$$

The rest of this section deals with the determination of $\{p\}$. Along the way we also find the values of $d_1, \ldots, d_k$.

The derivations here are based on a procedure first used in [24]; it is much simpler than the traditional Molien approach used, for example, in [21, 22].

Let $\{p\}$ now denote the (reducible in general) representation of H_k defined by the one-row Young tableau of p boxes acting in a space spanned by the polynomials of degree p in the components of $\square$, and let $[1^i]$ be the representation of H_k acting in the space spanned by the completely antisymmetric tensors of degree i in $\square$, defined by the one-column Young tableau of i boxes. In particular, $[1^1] = \square$.

Fundamental is the recurrence relation

$$\{p\} = \sum_{i=1}^{k} (-1)^{i-1} [1^i] \otimes \{p - i\}. \tag{4.4}$$

In starting the iteration of (4.4) we understand that $\{p\}$ is null for $p < 0$ and that $\{0\} = \chi_1$ is the invariant representation of H_k. Thus the first step of the iteration yields

$$\{1\} = [1^1] \otimes \{0\} = \Box \otimes \chi_1 = \Box.$$

The iteration of (4.4) to get higher $\{p\}$ is straightforward, except for modifications to take account of the removal of the denominator invariants corresponding to λ^{d_i} in (4.2). The modification is as follows. For p equal to a sum of m different integers d_i, i.e. $p = \sum_{j=1}^{m} d_{i_j}$, $1 \leqslant m \leqslant k$, a term $(-1)^{m-1}\chi_1$ is removed from $\{p\}$ before using (4.4) again to get $\{p+1\}, \ldots, \{p+k\}$. Each new d_i is found as the value of p at which a new invariant appears (one not removed by the modifications involving lower d_i's).

For most of the groups one encounters, the degrees of the invariants, appearing as the exponents d_i of the denominator terms in (4.2), are known. The above procedure then leads one readily to the desired generating function. If, however, the degrees of the denominator terms have to be determined during the iteration process, one faces a dilemma at each $\{p\}$ where the scalar representation χ_1 appears: Is it due to a new denominator term of degree p and as such should it be discarded for the subsequent iteration steps, or is it a true numerator term which has to be retained? Clearly the answer influences the results of subsequent steps. In most cases it is a new denominator term one finds. Exceptionally, there may also be a numerator term χ_1. A wrong decision concerning the dilemma often already leads to contradictions at the next step of the iteration, as we explain on an H_3 example at the end of section 6.

A general test for a numerator scalar at degree p consists of checking the dimension of $\{2p\}$, with terms involving powers of the denominator scalars as factors included. The dimension of the representation $\{2p\}$ can be calculated independently to be $(k+2p-1)!/((2p)!(k-1)!)$, but it will be greater by 1 if a numerator scalar at p was treated erroneously as a denominator scalar.

5. Generating functions for H_2

The character table for H_2 is given in table 1. The defining irreducible representation referred to as $\Box$ in section 4 is χ_3.

Table 1. The character table of $H_2 \simeq \mathcal{D}_5$.

x	$1a$	$2a$	$2b$	$5a$
#	1	2	2	5
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	$-\tau'$	$-\tau$	0
χ_4	2	$-\tau$	$-\tau'$	0

Table 2. The multiplicities of irreducible representations of H_2 in $\{p\}$. Zero entries are not shown.

p	0	1	2	3	4	5
χ_1	1					
χ_2						1
χ_3		1			1	
χ_4			1	1		

The representation $[1^2]$ needed in (4.4) is χ_2. The multiplicities of irreducible representations contained in $\{p\}$ are shown in table 2. The denominator scalars have degrees 2 and 5.

Using table 2 in (4.2) we find explicitly the desired generating function for the polynomial tensors based on χ_3:

$$\frac{\chi_1 + \lambda\chi_3 + \lambda^2\chi_4 + \lambda^3\chi_4 + \lambda^4\chi_3 + \lambda^5\chi_2}{(1-\lambda^2)(1-\lambda^5)} = \chi_1 + \lambda\chi_3 + \lambda^2(\chi_1+\chi_4) + \lambda^3(\chi_3+\chi_4) + \lambda^4(\chi_1+\chi_3+\chi_4) + \lambda^5(\chi_1+\chi_2+\chi_3+\chi_4) + \lambda^6(\chi_1+2\chi_3+\chi_4) + \cdots. \tag{5.1}$$

Here the interpretation of the terms of the expansion of the generating function into the power series is the standard one for a generating function. For example the term containing λ^6 indicates that the polynomials of degree 6 transform as the reducible representation $\chi_1 \oplus \chi_3 \oplus \chi_3 \oplus \chi_4$.

Very similar are the polynomial tensors based on χ_4 (i.e. where $\Box$ stands for χ_4 rather than for χ_3). Their generating function is given by (4.2) using a modified table 2 in which the rows opposite χ_3 and χ_4 are interchanged.

Table 2 was obtained using the recursion relations (4.4) and the value $p_{\max}$ of (4.3). Let us now illustrate how the recursion relations are used to obtain table 2.

Starting from the chosen representation $\chi_3 = \Box = [1^1]$, we find by standard methods also the antisymmetric part of $\chi_3 \otimes \chi_3$ to be $[1^2] = \chi_2$. Since χ_3 is two-dimensional, $[1^3] = 0$. Hence (4.4) has two terms,

$$\{p\} = [1] \otimes \{p-1\} + (-1)[1^2] \otimes \{p-2\}. \tag{5.2}$$

By definition $\{p\} = 0$ for $p < 0$ and $\{0\} = \chi_1$. Remembering that the terms χ_1 corresponding to the contribution to the power series (5.1) from the denominator terms are to be discarded, we find the following $\{p\}$ for $1 \leqslant p \leqslant p_{\max} = 5$:

$$\begin{aligned}
&\{1\} = \chi_3 \\
&\{2\} = \chi_3 \otimes \chi_3 - \chi_2 \otimes \chi_1 = \chi_1 + \chi_2 + \chi_4 - \chi_2 = \chi_1 + \chi_4 \Rightarrow \chi_4 \\
&\{3\} = \chi_3 \otimes \chi_4 - \chi_2 \otimes \chi_3 = \chi_4 \\
&\{4\} = \chi_3 \otimes \chi_4 - \chi_2 \otimes \chi_4 = \chi_3 \\
&\{5\} = \chi_3 \otimes \chi_3 - \chi_2 \otimes \chi_4 = \chi_1 + \chi_2 \Rightarrow \chi_2 \\
&\{6\} = \chi_3 \otimes \chi_2 - \chi_2 \otimes \chi_3 = 0 \\
&\{7\} = 0 - \chi_2 \otimes \chi_2 = -\chi_1 \Rightarrow 0.
\end{aligned} \tag{5.3}$$

Here the double arrow marks the places where χ_1 has been discarded. Clearly the recursion calculation terminates automatically when $p_{\max}$ has been reached. Table 2 is a concise way to present the information provided in (5.3).

6. Generating functions for H_3

The character table of H_3 is given by table 3. The generating function we wish to find is based on the three-dimensional representation χ_4. It is referred to as $\Box$ in section 4.

Antisymmetric parts of the powers of χ_4 are as follows:

$$[1] = \chi_4 \qquad [1^2] = \chi_3 \qquad [1^3] = \chi_2 \qquad [1^4] = 0 \qquad \ldots. \tag{6.1}$$

Table 3. The character table of H_3. The notation is described in section 7 in connection with the character table of H_4.

x	1a	2a	2b	2c	3a	6a	5a	5b	10a	10b
x^2	1a	1a	1a	1a	3a	3a	5b	5a	5b	5a
x^3	1a	2a	2b	2c	1a	2a	5b	5a	10b	10a
#	1	1	15	15	20	20	12	12	12	12
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	−1	1	−1	1	1	−1	−1
χ_3	3	3	−1	−1	0	0	τ	τ'	τ	τ'
χ_4	3	−3	−1	1	0	0	τ	τ'	$-\tau$	$-\tau'$
χ_5	3	3	−1	−1	0	0	τ'	τ	τ'	τ
χ_6	3	−3	−1	1	0	0	τ'	τ	$-\tau'$	$-\tau$
χ_7	4	4	0	0	1	1	−1	−1	−1	−1
χ_8	4	−4	0	0	1	−1	−1	−1	1	1
χ_9	5	5	1	1	−1	−1	0	0	0	0
χ_{10}	5	−5	1	−1	−1	1	0	0	0	0

Hence in this case there are three terms on the right-hand side of (4.4),

$$\{p\} = [1] \otimes \{p-1\} + (-1)[1^2] \otimes \{p-2\} + [1^3] \otimes \{p-3\}. \tag{6.2}$$

The multiplicities of irreducible components of all $\{p\}$ used in the recursion relations (4.4) are shown in the columns of table 4. The denominator invariants have degrees 2, 6, and 10.

Note that in table 4 the content of the line opposite each even χ_i is that of the preceding odd line with each degree p subtracted from 15.

Very similar are the generating functions for polynomial tensors based on the other three-dimensional representations of H_3, namely χ_3, χ_5, and χ_6. Similarly to the case of H_2, the representations $\{p\}$ required in (4.4) are easily read from a modified table 4. The modifications consist of the following.

(a) When χ_3 is $\square$ the resulting tensors χ_i all have odd i. For i odd the representation χ_i appears at a degree p for which either χ_i or χ_{i+1} is non-zero in table 4.

(b) When χ_5 is $\square$ the resulting tensors χ_i all have odd i. They appear at the following degrees.

χ_1: as shown in table 4 opposite χ_1 and χ_2;
χ_3: as shown in table 4 opposite χ_5 and χ_6;
χ_5: as shown in table 4 opposite χ_3 and χ_4;
χ_7: as shown in table 4 opposite χ_7 and χ_8;
χ_9: as shown in table 4 opposite χ_9 and χ_{10}.

(c) When χ_6 is $\square$, then χ_3, χ_4, χ_5, χ_6, appear at the degrees shown in table 4 for χ_5, χ_6, χ_3, χ_4, respectively; the other χ_i appear as shown in table 4.

Let us now iterate (6.2) in order to show how table 4 is obtained starting from the representations $\{0\} = \chi_1$, $\{1\} = \chi_4$, and putting $\{p\} = 0$ for $p < 0$. We have

$$\{2\} = \chi_4 \otimes \chi_4 - \chi_3 \otimes \chi_1 + 0 = \chi_1 + \chi_9 \Rightarrow \chi_9$$
$$\{3\} = \chi_4 \otimes \chi_9 - \chi_3 \otimes \chi_4 + \chi_2 \otimes \chi_1 = \chi_6 + \chi_8$$
$$\{4\} = \chi_4 \otimes (\chi_6 + \chi_8) - \chi_3 \otimes \chi_9 + \chi_2 \otimes \chi_4 = \chi_7 + \chi_9$$
$$\{5\} = \chi_4 \otimes (\chi_7 + \chi_9) - \chi_3 \otimes (\chi_6 + \chi_8) + \chi_2 \otimes \chi_9 = \chi_4 + \chi_6 + \chi_{10}$$

Table 4. The multiplicities of irreducible representations of H_3 appearing in $\{p\}$ of (4.2). Zero entries are not shown.

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
χ_1	1															
χ_2																1
χ_3							1				1				1	
χ_4		1				1				1						
χ_5									1		1		1			
χ_6				1		1		1								
χ_7					1		1		1				1			
χ_8				1				1		1		1				
χ_9			1		1		1		1		1					
χ_{10}						1		1		1		1		1		

$$
\begin{aligned}
\{6\} &= \chi_4 \otimes (\chi_4 + \chi_6 + \chi_{10}) - \chi_3 \otimes (\chi_7 + \chi_9) + \chi_2 \otimes (\chi_6 + \chi_8) \\
&= \chi_1 + \chi_3 + \chi_7 + \chi_9 \Rightarrow \chi_3 + \chi_7 + \chi_9 \\
\{7\} &= \chi_4 \otimes (\chi_3 + \chi_7 + \chi_9) - \chi_3 \otimes (\chi_4 + \chi_6 + \chi_{10}) + \chi_2 \otimes (\chi_7 + \chi_9) = \chi_6 + \chi_8 + \chi_{10} \\
\{8\} &= \chi_4 \otimes (\chi_6 + \chi_8 + \chi_{10}) - \chi_3 \otimes (\chi_3 + \chi_7 + \chi_9) + \chi_2 \otimes (\chi_4 + \chi_6 + \chi_{10}) \\
&= -\chi_1 + \chi_5 + \chi_7 + \chi_9 \Rightarrow \chi_5 + \chi_7 + \chi_9
\end{aligned}
$$

$\vdots$

$$
\begin{aligned}
\{14\} &= \chi_4 \otimes \chi_{10} - \chi_3 \otimes (\chi_5 + \chi_7) + \chi_2 \otimes (\chi_8 + \chi_{10}) = \chi_3 \\
\{15\} &= \chi_4 \otimes \chi_3 - \chi_3 \otimes \chi_{10} + \chi_2 \otimes (\chi_5 + \chi_7) = \chi_2 \\
\{16\} &= \chi_4 \otimes \chi_2 - \chi_3 \otimes \chi_3 + \chi_2 \otimes \chi_{10} = -\chi_1 \Rightarrow 0 \\
\{17\} &= -\chi_3 \otimes \chi_2 + \chi_2 \otimes \chi_3 = 0 \\
\{18\} &= \chi_2 \otimes \chi_2 = \chi_1 \Rightarrow 0.
\end{aligned}
$$

So we have derived table 4 and shown that the non-trivial iteration of (6.2) stops automatically at $p = p_{\max} = 15$.

Note also that during the iteration one determines the degrees d_i of the denominator terms of the generating function. Indeed, in the iteration the discarded χ_1 occurred at $p = 2$, 6, 10, $2+6+10 = 18$ while $-\chi_1$ was discarded at $p = 2+6 = 8$, $2+10 = 12$, $6+10 = 16$.

We now return to the dilemma mentioned at the end of section 4. The lowest example where the ambiguity about the numerator or denominator origin of the term χ_1 arises is the case of the generating function for H_3 based on the representation χ_3 instead of χ_4. Although such a generating function was obtained above by the modification rules (a)–(c), we could have calculated it directly by the iteration process. During the iteration a numerator scalar arises at the step $p = 15$. It is easy to see that an error has been made if it is removed as a denominator scalar: a term $-\chi_3$ remains at $p = 16$.

7. Generating functions for H_4

The character table of H_4 is table A1 in the appendix [27]. Because of its size the following shortcuts were adopted in comparison with the character tables 1 and 3. In the first column an irreducible representation χ_m is identified by its subscript m only. Negative signs are shown as overbars of the corresponding entries. The character values are either integers or of the form (a, b) and $(a, b)'$ where a and b are integers. These are to be read as

$$(a, b) = a + \tau b \qquad (a, b)' = a + \tau' b \qquad \tau = \tfrac{1}{2}(1 + \sqrt{5}) \qquad \tau' = \tfrac{1}{2}(1 - \sqrt{5}). \tag{7.1}$$

In the first line of the character table, headed by x, the conjugacy classes of the elements of H_4 are named. Each symbol contains the order of the elements of the class followed by a letter in alphabetic order within each class. The lines x^2, x^3, and x^5 give the conjugacy classes of the powers 2, 3, and 5 of x, respectively. The line # shows the number of elements in each conjugacy class.

The irreducible representation of H_4 referred to as $\square$ in section 4 is χ_3. The representations $[1^2]$, $[1^3]$, $[1^4]$ are, χ_7, χ_4, χ_2 respectively.

It is convenient to organize the tables of $\{p\}$ separately for p even and p odd because they contain mutually exclusive subsets of irreducible representations. Tables A2 and A3 of the appendix give the irreducible representations χ_i contained in $\{p\}$: table A2 for p even, table A3 for p odd. Note that the sum of the entries on a line χ_i is equal to the dimension of χ_i. The degrees of the denominator invariants are 2, 12, 20, and 30.

Certain irreducible representations occur in pairs, namely those with the following subscripts:

$$1, 2 \qquad 3, 4 \qquad 5, 6 \qquad 11, 12 \qquad 13, 14 \qquad 18, 19 \qquad 20, 21 \qquad 27, 28 \qquad 32, 32.$$

The degree at which each of these appears is that of its partner subtracted from 60. Those χ_i without a partner occur with the same multiplicity at degree p as at degree $60 - p$.

When $\square$ is χ_4, χ_5, or χ_6 instead of χ_3, we can give the generating function in terms of that for χ_3.

$\square = \chi_4$. The representations $[1^2]$, $[1^3]$, $[1^4]$ are χ_7, χ_3, χ_2, respectively. The rows of table A2 (p even) are unchanged. The rows of table A3 (p odd) corresponding to paired representations are interchanged; the others are unaffected.

$\square = \chi_5$. The representations $[1^2]$, $[1^3]$, $[1^4]$ are χ_8, χ_6, χ_2, respectively. The rows opposite the following pairs of χ_i are interchanged:

7, 8; 11, 13; 12, 14; 23, 24; 29, 30 in table A2;

3, 5; 4, 6; 16, 17; 25, 26 in table A3.

All other rows are not changed.

$\square = \chi_6$. The representations $[1^2]$, $[1^3]$, $[1^4]$ are χ_8, χ_5, χ_2, respectively. The following pairs are interchanged:

7, 8; 11, 13; 12, 14; 23, 24; 29, 30 in table A2;

3, 6; 4, 5; 16, 17; 18, 19; 25, 26; 31, 32 in table A3.

All other rows are not changed.

Acknowledgments

We are grateful for the support of the Natural Sciences and Engineering Research Council of Canada and the FCAR of Quebec. One of us (JP) would like to thank the Physics Department of McGill University for the hospitality extended to him during the past three years where most of this work was done and to the Fields Institute in Toronto where it was completed.

Table A1. The H_4 character and powermap table. For notation see section 7.

x	1*a*	2*a*	2*b*	3*a*	3*b*	4*a*	5*a*	5*b*	5*c*	5*d*	5*e*	6*a*	6*b*	10*a*	10*b*	10*c*	10*d*	10*e*	12*a*	15*a*
x^2	1*a*	1*a*	1*a*	3*a*	3*b*	2*a*	5*b*	5*a*	5*d*	5*c*	5*e*	3*a*	3*b*	5*a*	5*b*	5*c*	5*d*	5*e*	6*a*	15*b*
x^3	1*a*	2*a*	2*b*	1*a*	*aa*	4*a*	5*b*	5*a*	5*d*	5*c*	5*e*	2*a*	2*b*	10*b*	10*a*	10*d*	10*c*	10*e*	4*a*	5*b*
x^5	1*a*	2*a*	2*b*	3*a*	3*b*	4*a*	1*a*	1*a*	1*a*	1*a*	1*a*	6*a*	6*b*	2*a*	2*a*	2*a*	2*a*	2*a*	12*a*	3*a*
#	1	1	450	40	400	60	24	24	144	144	288	40	400	24	24	144	144	288	1200	400
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	4	$\bar{4}$	0	$\bar{2}$	1	0	$(0, \bar{2})$	$(0, \bar{2})'$	(1, 1)′	(1, 1)	$\bar{1}$	2	$\bar{1}$	(0, 2)′	(0, 2)	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{1})'$	1	0	(0, 1)
4	4	$\bar{4}$	0	$\bar{2}$	1	0	$(0, \bar{2})$	$(0, \bar{2})'$	(1, 1)′	(1, 1)	$\bar{1}$	2	$\bar{1}$	(0, 2)′	(0, 2)	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{1})'$	1	0	(0, 1)
5	4	$\bar{4}$	0	$\bar{2}$	1	0	$(0, \bar{2})'$	$(0, \bar{2})$	(1, 1)	(1, 1)′	$\bar{1}$	2	$\bar{1}$	(0, 2)	(0, 2)′	$(\bar{1}, \bar{1})'$	$(\bar{1}, \bar{1})$	1	0	(0, 1)′
6	4	$\bar{4}$	0	$\bar{2}$	1	0	$(0, \bar{2})'$	$(0, \bar{2})$	(1, 1)	(1, 1)′	$\bar{1}$	2	$\bar{1}$	(0, 2)	(0, 2)′	$(\bar{1}, \bar{1})'$	$(\bar{1}, \bar{1})$	1	0	(0, 1)′
7	6	6	$\bar{2}$	3	0	2	(3, 1)	(3, 1)′	(0, 2)′	(0, 2)	1	3	0	(3, 1)′	(3, 1)	(0, 2)	(0, 2)′	1	$\bar{1}$	(0, 1)
8	6	6	$\bar{2}$	3	0	2	(3, 1)′	(3, 1)	(0, 2)	(0, 2)′	1	3	0	(3, 1)	(3, 1)′	(0, 2)′	(0, 2)	1	$\bar{1}$	(0, 1)′
9	8	8	0	5	2	4	3	3	$\bar{2}$	$\bar{2}$	$\bar{2}$	5	$\bar{2}$	3	3	$\bar{2}$	$\bar{2}$	$\bar{2}$	1	0
10	8	$\bar{8}$	0	$\bar{4}$	2	0	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	4	$\bar{2}$	2	2	2	2	$\bar{3}$	0	1
11	9	9	1	0	0	$\bar{3}$	(0, 3)′	(0, 3)	(1, 1)	(1, 1)′	$\bar{1}$	0	0	(0, 3)	(0, 3)′	(1, 1)′	(1, 1)	$\bar{1}$	0	0
12	9	9	1	0	0	$\bar{3}$	(0, 3)′	(0, 3)	(1, 1)	(1, 1)′	$\bar{1}$	0	0	(0, 3)	(0, 3)′	(1, 1)′	(1, 1)	$\bar{1}$	0	0
13	9	9	1	0	0	$\bar{3}$	(0, 3)	(0, 3)′	(1, 1)′	(1, 1)	$\bar{1}$	0	0	(0, 3)′	(0, 3)	(1, 1)	(1, 1)′	$\bar{1}$	0	0
14	9	9	1	0	0	$\bar{3}$	(0, 3)	(0, 3)′	(1, 1)′	(1, 1)	$\bar{1}$	0	0	(0, 3)′	(0, 3)	(1, 1)	(1, 1)′	$\bar{1}$	0	0
15	10	10	2	4	$\bar{2}$	6	5	0	0	0	0	4	$\bar{2}$	5	5	0	0	0	0	$\bar{1}$
16	16	$\bar{16}$	0	$\bar{2}$	$\bar{2}$	0	$(\bar{2}, \bar{4})'$	$(\bar{2}, \bar{4})$	(0, 2)	(0, 2)′	1	2	2	(2, 4)	(2, 4)′	$(0, \bar{2})'$	$(0, \bar{2})$	$\bar{1}$	0	(0, 1)
17	16	$\bar{16}$	0	$\bar{2}$	$\bar{2}$	0	$(\bar{2}, \bar{4})$	$(\bar{2}, \bar{4})'$	(0, 2)′	(0, 2)	1	2	2	(2, 4)′	(2, 4)	$(0, \bar{2})$	$(0, \bar{2})'$	$\bar{1}$	0	(0, 1)′
18	16	$\bar{16}$	0	4	1	0	$\bar{4}$	$\bar{4}$	1	1	1	$\bar{4}$	$\bar{1}$	4	4	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$
19	16	$\bar{16}$	0	4	1	0	$\bar{4}$	$\bar{4}$	1	1	1	$\bar{4}$	$\bar{1}$	4	4	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$
20	16	16	0	4	1	0	$\bar{4}$	$\bar{4}$	1	1	1	4	1	$\bar{4}$	$\bar{4}$	1	1	1	0	$\bar{1}$
21	16	16	0	4	1	0	$\bar{4}$	$\bar{4}$	1	1	1	4	1	$\bar{4}$	$\bar{4}$	1	1	1	0	$\bar{1}$
22	18	18	2	0	0	$\bar{6}$	3	3	$\bar{2}$	$\bar{2}$	3	0	0	3	3	$\bar{2}$	$\bar{2}$	3	0	0
23	24	24	0	3	0	$\bar{4}$	$(\bar{3}, 4)$	$(\bar{3}, 4)'$	$(0, \bar{2})'$	$(0, \bar{2})$	$\bar{1}$	3	0	$(\bar{3}, 4)'$	$(\bar{3}, 4)$	$(0, \bar{2})$	$(0, \bar{2})'$	$\bar{1}$	$\bar{1}$	(0, 1)
24	24	24	0	3	0	$\bar{4}$	$(\bar{3}, 4)'$	$(\bar{3}, 4)$	$(0, \bar{2})'$	$(0, \bar{2})$	$\bar{1}$	3	0	$(\bar{3}, 4)$	$(\bar{3}, 4)'$	$(0, \bar{2})'$	$(0, \bar{2})$	$\bar{1}$	$\bar{1}$	(0, 1)′
25	24	$\bar{24}$	0	$\bar{6}$	0	0	$(\bar{4}, 6)$	$(\bar{4}, 6)'$	$(0, \bar{2})$	$(0, \bar{2})'$	$\bar{1}$	6	0	$(4, \bar{6})'$	$(4, \bar{6})$	(0, 2)′	(0, 2)	1	0	$\bar{1}$
26	24	$\bar{24}$	0	$\bar{6}$	0	0	$(\bar{4}, 6)'$	$(\bar{4}, 6)$	$(0, \bar{2})'$	$(0, \bar{2})$	$\bar{1}$	6	0	$(4, \bar{6})$	$(4, \bar{6})'$	(0, 2)	(0, 2)′	1	0	$\bar{1}$
27	25	25	1	$\bar{5}$	1	5	0	0	0	0	0	$\bar{5}$	1	0	0	0	0	0	$\bar{1}$	0
28	25	25	1	$\bar{5}$	1	5	0	0	0	0	0	$\bar{5}$	1	0	0	0	0	0	$\bar{1}$	0
29	30	30	$\bar{2}$	$\bar{3}$	0	$\bar{2}$	(0, 5)	(0, 5)′	0	0	0	$\bar{3}$	0	(0, 5)′	(0, 5)	0	0	0	1	$(0, \bar{1})$
30	30	30	$\bar{2}$	$\bar{3}$	0	$\bar{2}$	(0, 5)′	(0, 5)	0	0	0	$\bar{3}$	0	(0, 5)	(0, 5)′	0	0	0	1	$(0, \bar{1})'$
31	36	$\bar{36}$	0	0	0	0	6	6	1	1	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	0
32	36	$\bar{36}$	0	0	0	0	6	6	1	1	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	0
33	40	40	0	1	$\bar{2}$	4	$\bar{5}$	$\bar{5}$	0	0	0	1	$\bar{2}$	$\bar{5}$	$\bar{5}$	0	0	0	1	1
34	48	$\bar{48}$	0	6	0	0	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{6}$	0	2	2	2	2	2	0	1

Table A1. (Continued)

x	$15b$	$20a$	$20b$	$30a$	$30b$	$2c$	$2d$	$4b$	$6c$	$6d$	$10f$	$10g$	$10h$	$10i$
x^2	$15a$	$10a$	$10b$	$15b$	$15a$	$1a$	$1a$	$2b$	$3b$	$3b$	$5d$	$5d$	$5c$	$5c$
x^3	$5a$	$20b$	$20a$	$10a$	$10b$	$2c$	$2d$	$4b$	$2d$	$2c$	$10h$	$10i$	$10f$	$10g$
x^5	$3a$	$4a$	$4b$	$6a$	$6a$	$2c$	$2d$	b	$6c$	$6d$	$2d$	$2c$	$2d$	$2c$
#	480	720	720	480	480	60	60	1800	1200	1200	720	720	720	720
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
3	$(0, 1)'$	0	0	$(0, \bar{1})$	$(0, \bar{1})'$	2	$\bar{2}$	0	1	$\bar{1}$	$(0, 1)$	$(0, \bar{1})$	$(0, 1)'$	$(0, \bar{1})'$
4	$(0, 1)'$	0	0	$(0, \bar{1})$	$(0, \bar{1})'$	$\bar{2}$	2	0	$\bar{1}$	1	$(0, \bar{1})$	$(0, 1)$	$(0, \bar{1})'$	$(0, 1)'$
5	$(0, 1)$	0	0	$(0, \bar{1})'$	$(0, \bar{1})$	2	$\bar{2}$	0	1	$\bar{1}$	$(0, 1)'$	$(0, \bar{1})'$	$(0, 1)$	$(0, \bar{1})$
6	$(0, 1)$	0	0	$(0, \bar{1})'$	$(0, \bar{1})$	$\bar{2}$	2	0	$\bar{1}$	1	$(0, \bar{1})'$	$(0, 1)'$	$(0, \bar{1})$	$(0, 1)$
7	$(0, 1)'$	$(0, \bar{1})'$	$(0, \bar{1})$	$(0, 1)$	$(0, 1)'$	0	0	0	0	0	0	0	0	0
8	$(0, 1)$	$(0, \bar{1})$	$(0, \bar{1})'$	$(0, 1)'$	$(0, 1)$	0	0	0	0	0	0	0	0	0
9	0	$\bar{1}$	$\bar{1}$	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	$\bar{1}$	$\bar{1}$	0	0	0	0	0	0	0	0	0
11	0	$(0, \bar{1})'$	$(0, \bar{1})$	0	0	3	3	$\bar{1}$	0	0	$(0, 1)'$	$(0, 1)'$	$(0, 1)$	$(0, 1)$
12	0	$(0, \bar{1})'$	$(0, \bar{1})$	0	0	$\bar{3}$	$\bar{3}$	1	0	0	$(0, \bar{1})'$	$(0, \bar{1})'$	$(0, \bar{1})$	$(0, \bar{1})$
13	0	$(0, \bar{1})$	$(0, \bar{1})'$	0	0	3	3	$\bar{1}$	0	0	$(0, 1)$	$(0, 1)$	$(0, 1)'$	$(0, 1)'$
14	0	$(0, \bar{1})$	$(0, \bar{1})'$	0	0	$\bar{3}$	$\bar{3}$	1	0	0	$(0, \bar{1})$	$(0, \bar{1})$	$(0, \bar{1})'$	$(0, \bar{1})'$
15	$\bar{1}$	1	1	$\bar{1}$	$\bar{1}$	0	0	0	0	0	0	0	0	0
16	$(0, 1)'$	0	0	$(0, \bar{1})$	$(0, \bar{1})'$	0	0	0	0	0	0	0	0	0
17	$(0, 1)$	0	0	$(0, \bar{1})'$	$(0, \bar{1})$	0	0	0	0	0	0	0	0	0
18	$\bar{1}$	0	0	1	1	4	$\bar{4}$	0	$\bar{1}$	1	1	$\bar{1}$	1	$\bar{1}$
19	$\bar{1}$	0	0	1	1	$\bar{4}$	4	0	1	$\bar{1}$	$\bar{1}$	1	$\bar{1}$	1
20	$\bar{1}$	0	0	$\bar{1}$	$\bar{1}$	4	4	0	1	1	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
21	$\bar{1}$	0	0	$\bar{1}$	$\bar{1}$	$\bar{4}$	$\bar{4}$	0	$\bar{1}$	$\bar{1}$	1	1	1	1
22	0	$\bar{1}$	$\bar{1}$	0	0	0	0	0	0	0	0	0	0	0
23	$(0, 1)'$	1	1	$(0, 1)$	$(0, 1)'$	0	0	0	0	0	0	0	0	0
24	$(0, 1)$	1	1	$(0, 1)'$	$(0, 1)$	0	0	0	0	0	0	0	0	0
25	$\bar{1}$	0	0	1	1	0	0	0	0	0	0	0	0	0
26	$\bar{1}$	0	0	1	1	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	5	5	1	$\bar{1}$	$\bar{1}$	0	0	0	0
28	0	0	0	0	0	$\bar{5}$	$\bar{5}$	$\bar{1}$	1	1	0	0	0	0
29	$(0, \bar{1})'$	$(0, 1)$	$(0, 1)'$	$(0, \bar{1})$	$(0, \bar{1})'$	0	0	0	0	0	0	0	0	0
30	$(0, \bar{1})$	$(0, 1)'$	$(0, 1)$	$(0, \bar{1})'$	$(0, \bar{1})$	0	0	0	0	0	0	0	0	0
31	0	0	0	6	$\bar{6}$	0	0	0	$\bar{1}$	1	$\bar{1}$	1	0	0
32	0	0	0	$\bar{6}$	6	0	0	0	1	$\bar{1}$	1	$\bar{1}$	0	0
33	1	$\bar{1}$	$\bar{1}$	1	1	0	0	0	0	0	0	0	0	0
34	1	0	0	$\bar{1}$	$\bar{1}$	0	0	0	0	0	0	0	0	0

Table A2. The multiplicities of irreducible representations in $\{p\}$, with p even, appearing in the numerator of the H_4 generating function based on χ_3. Zero entries are not shown.

p	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	68	60
χ_1	1																														
χ_2																															1
χ_7							1				1					2					1				1						
χ_8											1		1			2			1		1										
χ_9							1				1		1			2			1		1				1						
χ_{11}				1		1		1	1		1		1	1		1		1													
χ_{12}														1		1		1	1		1		1	1		1		1			
χ_{13}		1				1	1			1	1	1			1	1				1											
χ_{14}												1				1	1			1	1	1			1	1				1	
χ_{15}							1				2		1			2			1		2				1						
χ_{20}				1	1		1	1	1	2		1	2	1	1		1	1	1			1									
χ_{21}										1			1	1	1		1	1	2	1		2	1	1	1		1	1			
χ_{22}						1		1		1	1	1	1	1	1	2	1	1	1	1	1	1		1		1					
χ_{23}							1	1		2	1	2	1	1	2	2	2	1	1	2	1	2		1	1						
χ_{24}				1				1	1	1	1	1	2	2	1	2	1	2	2	1	1	1	1	1				1			
χ_{27}			1		1	1	1	1	2	1	2	2	1	2	2	1	2	1	1	1	1		1								
χ_{28}									1		1	1	1	1	2	1	2	2	1	2	2	1	2	1	1	1	1		1		
χ_{29}						1	1	1		1	2	3	1	1	3	2	3	1	1	3	2	1		1	1	1					
χ_{30}						1		1	2	1	2	1	2	3	1	2	1	3	2	1	2	1	2	1		1					
χ_{33}					1		1	1	2	2	1	2	3	3	3	2	3	3	3	2	1	2	2	1	1		1				

Table A3. The multiplicities of irreducible representations in $\{p\}$, with p odd, appearing in the numerator of the H_4 generating function based on χ_3. Zero entries are not shown.

p	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59
χ_3	1					1				1					1															
χ_4																1					1				1					1
χ_5				1			1		1			1																		
χ_6																			1			1		1			1			
χ_{10}							1			1		1			1	1			1		1			1						
χ_{16}							1		1	1		2		2	1	1	2		2		1	1		1						
χ_{17}						1	1			1	2	1			2	2			1	2	1			1	1					
χ_{18}		1			1	1	1		1	1	2	1		2	1	1	1		1	1										
χ_{19}											1	1		1	1	1	2		1	2	1	1			1	1			1	
χ_{25}				1			1		2	1		2	2	2	1	1	2	2	2		1	2		1			1			
χ_{26}						1	1			2	2	1	2		3	3		2	1	2	2			1	1					
χ_{31}			1	1	1	1	1	3	2	2	2	2	4	2	2	2	2	3	1	1	1	1	1							
χ_{32}								1	1	1	1	1	3	2	2	2	2	4	2	2	2	2	3	1	1	1	1	1		
χ_{34}					1	1	1	2	2	2	3	3	2	4	3	3	4	2	3	3	2	2	2	1	1	1				

References

[1] DiVincenzo D P, Mosseri R, Brodsky M H and Sadoc J F 1984 Long range structural and electronic coherence in amorphous semiconductors *Phys. Rev.* B **29** 5934–6

[2] DiVincenzo D P and Brodsky M H 1985 Polytope-like order in random network model of amorphous semiconductors *J. Non-Crystall. Solids* **77 & 78** 241–4

[3] Mosseri R, DiVincenzo D P, Sadoc J F and Brodsky M H 1985 Polytope model and the electronic and structural properties of amorphous semiconductors *Phys. Rev.* B **32** 3974–4000

[4] Brodsky M H, DiVincenzo D P and Mosseri R 1985 A structural basis for electronic coherence in amorphous Si and Ge *Proc. of 17th Int. Conf. on the Physics of Semiconductors* pp 803–6

[5] Fradkin M A 1987 Phonon spectrum of metallic glasses in an icosahedral model *Sov. Phys.–JETP* **66** 822–8

[6] DiVincenzo D P 1988 Nonlinear optics as a probe of chiral ordering in amorphous semiconductors *Phys. Rev.* B **37** 1245–61

[7] Bul'enkov N A 1991 Possible role of hydration as the leading integration factor in the organization of biosystems at different levels of their hierarchy *Biophysics* **36** 181–244

[8] Elser V and Sloane N J A 1987 A highly symmetric four-dimensional quasicrystal *J. Phys. A: Math. Gen.* **20** 6161–8

[9] Moody R V and Patera J 1993 Quasicrystals and icosians *J. Phys. A: Math. Gen.* **26** 2829–53

[10] Patera J 1997 Non-crystallographic root systems and quasicrystals *Proc. NATO ASI, Waterloo, 1995* ed R V Moody (Dordrecht: Kluwer) to be published

[11] Scherbak O P 1988 Wavefronts and reflection groups *Russian Math. Surveys* **43:3** 149–94

[12] Gross D J, Harvey J A, Martinec E and Rhom R 1985 Heterotic string *Phys. Rev. Lett.* **54** 502–5

[13] Hiller H 1982 *Geometry of Coxeter Groups* (Boston, MA: Pitman)

[14] Humphreys J E 1990 *Reflection Groups and Coxeter Groups* (Cambridge: Cambridge University Press)

[15] Deodhar V V 1982 Root system of a Coxeter group *Comm. Algebra* **10** 611–30

[16] Molien T 1897 *Sitzungber. König. Preuss. Acad. Wiss.* p 1152

[17] McLellan A G 1974 Invariant functions and homogeneous bases of representations of the crystal point groups, with applications to thermodynamic properties of crystals under strain *J. Phys. C: Solid State Phys.* **7** 3326–40

[18] Bethe H 1929 Termaufspaltung in Kristallen *Ann. Phys.* **3** 133–208

[19] von der Lage F C and Bethe H 1947 A method for obtaining electronic eigenfunctions and eigenvalues in solids with an application to sodium *Phys. Rev.* **71** 612–22

[20] Michel L 1977 Invariants polynomioux des groupes de symetrie moleculaire et cristallographique *Group Theoretical Methods in Physics (Proc. of the 5th Int. Coll.)* ed B Kolman and R T Sharp (New York: Academic) pp 75–91

[21] Patera J, Sharp R T and Winternitz P 1978 Polynomial irreducible tensors for point groups *J. Math. Phys.* **19** 2362–76

[22] Desmier P and Sharp R T 1979 Polynomial tensors for double point groups *J. Math. Phys.* **20** 74–82

[23] Patera J and Sharp R T 1980 Generating functions for plethysms of continuous and finite groups *J. Phys. A: Math. Gen.* **13** 397–416

[24] Desmier P E, Patera J and Sharp R T 1982 Analytic $SU(3)$ states in a finite subgroup basis *J. Math. Phys.* **23** 1383–8

[25] Cummins C J and Patera J 1988 Polynomial icosahedral invariants *J. Math. Phys.* **29** 1736–45

[26] Champagne B, Kjiri M, Patera J and Sharp R T 1995 Description of reflection generated polytopes using decorated Coxeter diagrams *Can. J. Phys.* **73** 566–84

[27] Grove L C 1974 The characters of the hecatonicoicosahedral group *J. reine angew. Math.* **265** 160–9